A Classical Electromagnetic Model for Thermal Emission from Ohmic Materials

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***Abstract*— Rytov's impressed currents have been for 70 years the basis of modern radiometry. This paper clarifies that they are affected by an important unjustified assumption: they are characterized by zero cross correlation. Here, resorting to a rigorous Electromagnetic analysis we propose an expansion for the electromagnetic fields in lossy media in terms of quasi-orthonormal functions that verify Maxwell's equations in an infinite lossy medium. These functions naturally define the independent currents that can co-exist in a lossy medium: the degrees of freedom of the medium. An entirely classic model, with Johnson like thermal sources, emerges to estimate the radiated power from finite samples of ohmic materials. A high frequency cut off in thermal radiation is found to be dependent from the conductivity and the related scattering time. This model leads to a much more comprehensible description of the radiometry radiation mechanism, without involving Plank's radiation law.**

***Index Terms*—Thermal noise, radiometry, degrees of freedom**

I. Introduction

The present status of the radiometric science is fairly puzzling. The field entirely relies on the *emissivity* of the bodies, which expresses the power emitted in comparison to the power emitted by an universal Black Body (BB) radiator, prescribed by Planck's law[1]. The majority of the experimental verifications of the validity of Planck's radiation law are 100 years old [2]. The part that is puzzling is that, apart from these old ones, we could not find any experiment that relates directly the power radiated by a general piece of material body to the BB spectrum. The mediation via the spectrum of the emissivity is mandatory and crucial for virtually all samples. The BB, is not even an upper limit since there are super Planckian radiators. Admittedly, the BB radiator does not exist in nature. There are, instead, [3] artificially engineered black body sources that are developed for calibration purposes, synthesized to have spectral behaviour that matches that of an ideal black body.

It seems that the problem resides in the fact that Planck's radiation law is not a classic electromagnetic description of radiation. This latter would requires Maxwell's equations and information on the properties of the material such as its conductivity. The only widely accepted electromagnetic model for thermally generated radiation was proposed by Rytov [4] in the '60. It was an hybrid classic-quantum model that relied in solving Maxwell's equations classically with a forcing term represented by *impressed currents* whose spectrum was quantum corrected for the higher frequencies by introducing the number of photons ad hoc [1]. The theory was based on the fluctuation dissipation theorem (FDT).

This lack of comforting experimental validations of the emission theory presented by Rytov is due to the fact that his source currents are affected by a fundamental problem that, somehow, has gone unnoticed for 70 years. In every point of the volume the impressed currents are assumed to be statistically independent from the impressed currents in any other point. In other words Rytov's currents are taken with radius of correlation equal to zero. Initially, in [5] from 1950's, Rytov proposed this as an assumption. Later on, comforted by the success in the Physics community [6], he proposed a demonstration in his 1989 book [4], that appears to be at least inconclusive. While this is the first problem with Rytov's currents, the second is the impossibility to provide a uniquely defined discretization for them, and finally their unmotivated frequency dependence from the Planck's photon number, rather the property of the material hosting them.

To solve these problems in section III a new proposal for currents to be used as sources in Maxwell's equation to investigate the emission from thermally excited ohmic sources is proposed. Modal functions that are solutions of Maxwell' s equations and quasi-orthogonal are taken in order to construct the source currents that represent the independent degrees of freedom of the material. These currents are, electromagnetically, independent as they are associated to locations in the volume, $\vec{r}_i$, that are not close in terms of the Green's function of the medium. Currents that are very close present high mutual coupling and consequently are correlated. The number of degrees of freedom, independent sources, per unit of volume is dictated by the penetration depth in the investigated material. This is, in turn, dependent from the conductivity considered, and its frequency dispersion that is described via the Drude's model [7]-[9].

The new currents predicts "Planckian like" high frequency frequency dispersion: the typical exponential attenuation is found. The frequency dispersion qualitatively match the predictions from Rytov's currents for conducting powders at densities used in absorbers. However, they are very different for other materials such as semi-conductors materials. The exponential attenuation of the currents amplitude is particularly important. That is because, in absence of this frequency dispersion, it was recently shown, [10] that radiometric currents would predict a asymptotically flat radiation spectrum. Now a full wave radiometric tool predicts as in [10] predicts the right frequency decay. This will be possibly addressed in a companion paper in this conference..

II. RYTOV'S CURRENTS

In [4] Rytov provided a set of electric currents that could be taken as impressed sources for Maxwell's equations in order to estimate the power radiated in free space as a result of thermal agitation. The theory was based on the fluctuation dissipation theorem (FDT) which was already used by Nyquist [11] to justify the very good experimental results of Jhonson [12] in characterizing the thermal noise sources in circuits. Bekefi [13], first interpreted the contribution from Rytov's which was fairly cryptic, and put it in words that could be better understood by the western scientific society. For Rytov and for all following literature including the modern one, [14]-[19] calculating the radiated power has been calculating directly the power radiated by the impressed currents using the relevant Green's function, when it was known.

The currents were specified in the frequency domain, in every point of the considered volume, by providing their correlation and postulating that the currents in each observation point were uncorrelated with the ones in neighboring ones. The specific expression is found in many references, each time is a slightly different form, and thus here we will present them in a form that is congruent with the notations typically used in electrical engineering to facilitate the readers. Indicating with << , >> the operation of co/cross correlation the currents centered at $\vec{r}_i$ and $\vec{r}_j$ are characterized by

$$<< \vec{J}_{Rytov}(\vec{r}_i), \vec{J}_{Rytov}(\vec{r}_j) >>= \frac{4hf}{e^{\frac{hf}{k_B T}}-1} Re\{\sigma\}\delta(\vec{r}_i - \vec{r}_j) \quad (1)$$

Here, k_B and h are the Boltzman and Planck's constants respectively, f indicates the frequency, T the temperature in Kelvin and σ the conductivity of the material hosting the currents. The conductivity in the frequency domain is complex. The retaining of only the real part of the conductivity, $Re\{\sigma\}$, and the multiplication per 4 were proposed as necessary steps in order for the definition of the currents in (1) to be taken as extension to a continuum of currents of the Fluctuation Dissipation Theorem. This was the well established base for the Johonson Noise discrete current sources used in electrical circuits. Hereinafter, the three clear problems with Rytov currents will be highlighted, to clarify the need for a new proposal.

A. *Problematic lack of correlation*

The fact that possible currents in source points $\vec{r}_j$ and $\vec{r}_i$ extremely close to each other can be uncorrelated and still be valid sources for the Maxwell's equation was the main take home message that the international community got from Bekefi [13], based on the findings of Rytov. Initially, in [5], the currents were only postulated to be uncorrelated, as in (1) and a discussion about the problems, that this would create was presented in the concluding chapter of [5]. However later on in Chapters 3.3 and 3.4 of [4], Rytov proposed an attempt to prove that that the cross correlations between currents that are not co-located would be zero. However, it appears to only demonstrate the completeness of possible expansions of solutions of Maxwell's equations. In pages 361-363 of [6] there is also an attempt to justify the cross correlation radius being equal to zero by using functions chosen as

$$\phi_j^L(\vec{r}) = rect(\vec{r} - \vec{r}_j, \epsilon) \text{ where } rect(\vec{r}, \epsilon) = \begin{matrix} 1 \;\forall\; |\vec{r}| < \epsilon \\ 0 \;\forall\; |\vec{r}| > \epsilon \end{matrix} \quad (2)$$

to expand both the fields and the sources. However, even if the expansion functions $\phi_j^L(\vec{r})$ in (2) are orthogonal they do not verify Maxwell's equations. Overall, there does not seem to be a proof that the cross correlation radius of source currents verifying Maxwell's equation can be equal to zero

B. *Problematic Discretization*

Apart from the theoretical basis a second important problem with Rytov's currents is their discretization. To facilitate the calculations necessary to derive the Poynting vector radiated by the impressed currents in the far field, Rytov suggested to use reciprocity which he indicated as the generalized Kirkhoff law[19]. The use of reciprocity allowed the scientific community, [14]-[16] to mention a few, to study the absorption properties and interpret them as emission properties. Thus the community largely avoided the question of how one would discretize Rytov's currents.

In [19] the Rytov currents have been discretized to estimate the emission from a finite body. There the Rytov's currents have been assumed to be distributed volumetrically and be constant over small cubic discretization domains. These currents, in the frequency domain, had constant amplitude in each domain but a stochastic phase, thus could be expressed as:

$$\vec{J}_{Rytov}(\vec{r}) = \lim_{\epsilon\to 0} \sum_{i=1}^{N} I_{Rytov}^{i} \frac{1}{\epsilon^2}\, rect(\vec{r} - \vec{r}_i, \epsilon^3)\hat{p}_i \quad (3)$$

with

$$I_{Rytov}^{i} = \sqrt{\frac{4hf}{e^{\frac{hf}{k_B T}}-1} Re\{\sigma\}\epsilon}\; e^{j\phi(\vec{r}_i)} \quad (4)$$

where $rect(\vec{r} - \vec{r}_i, \epsilon^3)$ is a function that is equal to one only in a cube of side ϵ centered in $\vec{r}_i$. The normalization $\frac{1}{\epsilon^2}$ is guarantee that a unitary amplitude of the current I_{Rytov}^{i} implies 1 ampere of net current along $\hat{p}_i$ in the cube. The phase ϕ can be assumed to be uniformly distributed between $\in (0, 2\pi)$. This reflects the fact that the source currents have zero value in time, but non zero average moduli. The problem with this representation is that according to Rytov's description, the limit can be taken for $\epsilon \to 0$. Equivalently N in (3) could tend to infinity, also if a volume if finite. In fact, since Rytov currents (1) are characterized by a non physical cross correlation distance equal to zero, there can be infinite sources of finite energy on a finite volume! Rytov knew this in his first work [4], and he commented in the conclusions that, as long as his currents were only used to evaluate the radiated power outside the volume considered, this should not have been a problem.

C. Problematic Frequency Dependence

A final problem with Rytov currents is their frequency dependence. Apart from the one associated to the conductivity, the explicit frequency dependence in (1) is the one of the Black Body radiation, accounting for Planck's law [1]. While this is certainly congruent with all existing literature, in eqs. (1this frequency dependence is added on without any dedicated justification, just like the Planck's law was also added on by Nyquist [11] to the experimentally measured Jhonson noise[12].

III. Proposal for New Currents

In this section a new set of currents to be used as sources in Maxwell's equation to investigate the emission from thermally excited ohmic sources is proposed. The justification for these currents, completely classical, is presented in section IV.

A. New Currents: Operative Form

The new currents, $\vec{J}_{Classic}(\vec{r})$, that can replace the currents from Rytov in (1), are directly discretized. As shown in Fig.1 the currents are expressed as the superposition of equal amplitude contributions emerging from $\mathrm{N_{DoF}}(f)$ different locations, each associated to a degrees of freedom of the waves in the volume, Vol, in the material considered at frequency f.

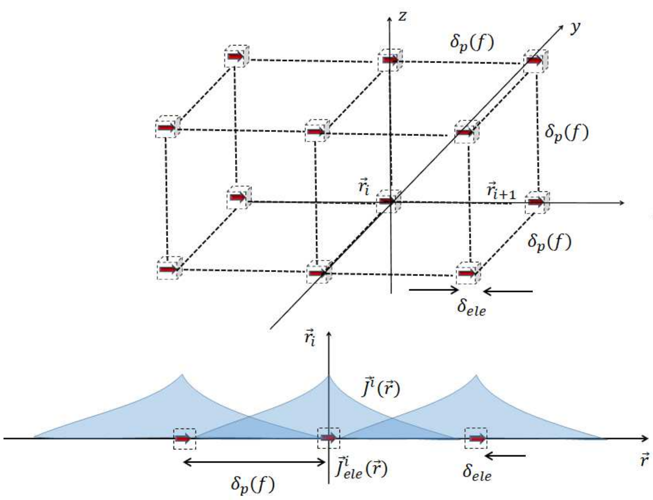

Fig.1 Top: the originating elementary cubic dipoles of side δ_{ele}, distributed with periods equal to the penetration depth in the medium $\delta_p(f)$ at the considered frequency. Bottom: the corresponding distribution of the electric fields and conduction currents, $\vec{J}^i(\vec{r})$.

The actual expression is:

$$\vec{J}_{Classic}(\vec{r}) = \sqrt{4k_BT\, Re\{\sigma\}\delta} \sum_{i=1}^{N_{DoF}} \vec{J}^i_{ele}(\vec{r}) \tag{5}$$

where $\vec{J}^i_{ele}(\vec{r})$, with $i = 1, N_{DoF}$, represent a set of currents that are electromagnetically orthogonal in the volume. The currents are dipoles assumed to be cubic, of side δ_{ele} electrically small, and centred at $\vec{r}_i$, and oriented along $\hat{x}$, $\hat{y}$ or $\hat{z}$. The current distribution of the originating dipoles can be described as:

$$\vec{J}^i_{ele}(\vec{r}) = \frac{1}{\delta^2_{ele}} rect[\vec{r}, \vec{r}_i, \delta^3_{ele}]\hat{p}_i \tag{6}$$

with $\hat{p}_i$ being any of the one of the Cartesian unit vectors. In (5) $\mathrm{N_{DoF}}(f)$ can be expressed as

$$\mathrm{N}_{DoF}(f) \equiv \frac{\mathrm{Vol}}{\frac{4}{3}\pi\delta^3_p(f)} \tag{7}$$

where $\delta_p(f)$ is the penetration depth, distance at which the mutual coupling between two elementary currents can be assumed to be zero (next section). This choice of the currents guarantees that the energy available for radiation or losses in the volume is equal to

$$E_{avail} = N_{DoF}\, k_BT \tag{8}$$

B. Drude's model

The penetration depth expressed as function of the imaginary part of the effective propagation constant in the lossy medium: $\delta_p(f) = -1/\,Im\{k_{eff}(f)\}$. This latter can be expressed as $k_{eff}(f) = k_0\sqrt{\epsilon_{r,eff}}$ with k_0 is the free space propagation constant and $\epsilon_{r,eff}$ the effective relative dielectric constant

$$\epsilon_{r,eff} = \sqrt{\epsilon_{rd} - \frac{j\sigma}{2\pi f\epsilon_0\epsilon_{rd}}} \tag{9}$$

and σ is the complex conductivity as predicted by Drude's model:

$$\sigma = \frac{\sigma_{qs}}{1+j\omega\tau} \tag{10}$$

The penetration depth is shown in Fig.2 for a bad metal characterized by $n_{ele} = 10^{24}, \tau_s = 7\ 10^{-15}$, comparison with free space wavelength. The frequency dependence of the expression of $N_{DoF}(f)$, normalized to its maximum is presented in Fig.3A, for the same bad conductor of Fig.2. It is compared to the frequency dependence of the function, $\frac{hf}{e^{\frac{hf}{k_BT}}-1}$, also normalized to its maximum, for a case of $T = 300\,K$. The normalized frequency dependence is exponentially decaying for both curves at the higher frequencies, while it presents important differences at very low frequencies. The curves are presented in normalized form, because while $\frac{hf}{e^{\frac{hf}{k_BT}}-1}$ is essentially unity at low frequencies and then smaller, N_{DoF} is in fact a very large number. It is equal to the number of many degrees of freedom per cubic meters, associated to the electric currents.

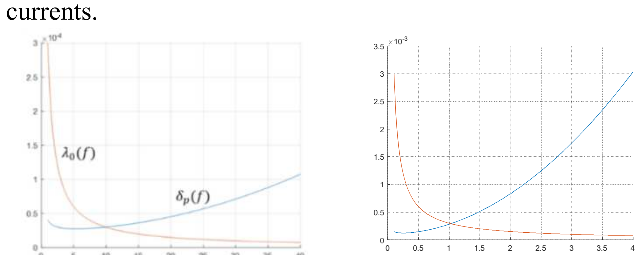

Fig.2 penetration depth for a bad metal characterized by $n_{ele} = 10^{24}, \tau_s = 7\ 10^{-15}$, comparison with free space wavelength

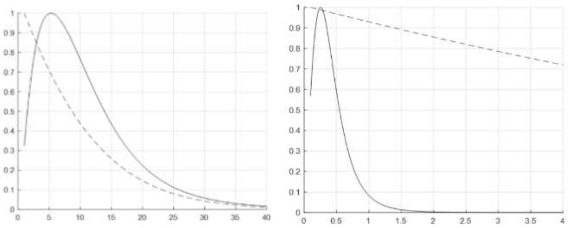

Fig.3 $N_{DoF}(f)$, normalized to its maximum compared to the normalized frequency dependence of the function, $hf\left[e^{\frac{hf}{k_BT}}-1\right\}^{-1}$. A) For the of the bad metal in Fig.1A. B) For the case of the semiconductor in Fig1B

In Fig.3B, presents the same information as Fig.3A but for a low resistivity silicon doped with $n = 8\,10^{21}$ electrons per cubic meter is shown. The scattering time in silicon is much longer, in the order of $\tau_{silicon} \approx 210^{-13}$ and accordingly $n_{DoF}(f)$ peaks at much lower frequencies.

IV. Justification of the New Currents

The classical currents in (5) are expanded in terms of the individual currents, in (6), that by construction verify Maxwell's equations. If these currents are orthonormal in space (or "quasi" as shown in the next subsection), they can be associated to independent degrees of freedom of *EM* waves in ohmic materials. According to the Equipartition Theorem, each independent degree of freedom in a Thermodynamic system is associated to energy k_BT. The incoherent superposition of uncorrelated noise sources is done routinely in Electrical engineering for fluctuating currents and voltages. Here it is suggested to extend the procedure of Jhonson sources presented in [8] to the multiple discrete lumped components that can be identified in Fig.1. The excited cubic block in Fig1, when considered embedded in an infinite medium, is characterized by the constitutive relations of the medium

$$\vec{j} = \sigma\vec{e} \tag{11}$$

If in the volume the current is expressed as in eq. the electric field, that must have the same distribution is can also be represented by the same uniform distribution on the cube that can be described as

$$\vec{e}^{\,i}_{ele}(\vec{r}) = \frac{v_{ele}}{\delta_{ele}} rect[\vec{r}, \vec{r}_i, \delta^3_{ele}]\hat{p}_i \tag{12}$$

Inserting the constitutive relation in (11) leads to

$$\mathrm{i}_{\mathrm{ele}} \frac{1}{\sigma\delta_{ele}} = v_{ele} \tag{13}$$

Which means one can define a lumped impedance to the elementary impressed currents

$$Z_\Omega = \frac{v_{ele}}{\mathrm{i}_{\mathrm{ele}}} = \frac{\rho}{\delta_{ele}} \tag{14}$$

A current and voltage couple, associated to the a lumped component, Z_Ω, with resistivity R_Ω, can provide a certain amount of spectral energy to a load, R_{load}, connected to it. If the load $R_{load} = R_\Omega$ then the energy rendered available is actually k_BT. Accordingly the required equivalent Thevenin voltage source is $V_{Th} = \sqrt{4k_BTR_\Omega}$. If instead of using a Thevenin representation, one was using a Norton representation, the equivalent current source would have to be $I_{Nort} = \frac{\sqrt{4k_BTR_\Omega}}{|Z_\Omega|}$. If a volume contains N_{DoF} independent degrees of freedom, each with voltage spectral content V_{Th} as in (5), the spectral energy available to the connected loads would be

$$E_{avail} = N_{DoF}\, k_BT \tag{15}$$

A. Quasi Orthonormal Modal expansion

For the sources in (5) to be Jhonson like current sources it is sufficient to demonstrate that the reaction between the currents $\vec{j}^i$ and the fields $\vec{e}^j$ due to $\vec{j}^j$ is equal to zero, or almost, i.e.

$$<\vec{j}^i, \vec{e}^j> \approx 0\ \forall |\vec{r}_i - \vec{r}_j| > \delta_t(f) \tag{16}$$

where the symbol $<,>$ expresses

$$<\vec{f}, \vec{g}> = \iiint_{-\infty}^{\infty} \vec{f}(\vec{r}) \cdot \vec{g}^*(\vec{r}) d\vec{r} \tag{17}$$

and is adopted to evaluate orthogonality in a volume. In Fig. 1, it is apparent that the domain of the electric fields radiated by the impressed currents $\vec{J}^i_{ele}$, are distributed on domains much larger than the cubic domains of the generating dipoles $\vec{J}^i_{ele}(\vec{r})$: again this latter was selected, with length δ_{ele} small with respect to the variability of the fields. The electric field radiated by each of the currents in (6) is given by the spatial convolution between the elementary current and the appropriate lossy medium Green's function:

$$\vec{e}^{\,i}(\vec{r}) = \bar{\bar{g}}(\vec{r}, \vec{r}_i, \sigma) * \vec{J}^i_{ele}(\vec{r}) \tag{18}$$

The integrals $<\vec{j}^i, \vec{e}^j>$ in (16) for $i \neq j$ would need to be computed numerically. However, to obtain the estimation of when they are negligible one can simply realize that all the Green's function's components are in fact proportional to the exponential associated to the propagation constant

$$\bar{\bar{g}}(\vec{r}, \vec{r}_i, \sigma) \propto e^{-jk_{eff}|\vec{r}-\vec{r}_i|} \tag{19}$$

So that, in turn the mutual couplings are proportional to

$$<\vec{j}^i, \vec{e}^j> \propto \iiint_{-\infty}^{\infty} e^{-jk_{eff}|\vec{r}-\vec{r}_j|} d\vec{r} \tag{20}$$

This proportionality means that only when the two originating elementary dipoles are at large distances with respect to the attenuation constant, see Fig.1 Bottom, the value of their corresponding coupling $<\vec{j}^i, \vec{e}^j>$ will be negligible. There cannot be a hard boundary between negligible and not negligible mutual coupling. Here, it is proposed to assume that $e^{-Im\{k_{eff}\}|\vec{r}_i-\vec{r}_j|} < e^{-1}$ which leads to

$$<\vec{j}^i, \vec{e}^j> \approx 0\ \ for\ |\vec{r}_i - \vec{r}_j| > \frac{-1}{Im\{k_{eff}(f)\}} = \delta_p(f) \tag{21}$$

where $\delta_p(f)$ is typically indicated as penetration depth. Accordingly one can assume that

$$< \vec{J}^i, \vec{e}^j > \approx 0 \; \forall |\vec{r}_i - \vec{r}_j| > \delta_t(f) \quad (22)$$

where $\delta_t(f)$, the threshold distance, is taken $\delta_t(f) = \delta_p(f)$ to guarantees that $\vec{J}^i, \vec{e}^j$ are quasi-orthogonal modal functions for $i \neq j$.

B. Comments on the lack of a rigorous threshold

The lovers of rigour could be disturbed by the approximate symbol in (21) and (22). To this regard one could use $\delta_t(f) = 2\,\delta_p(f)$ and the frequency dependence of the number of degrees of freedom would not change. Only the actual number would change. Evidently there is an arbitrariness in the choice of the threshold $\delta_t(f)$. This is the same situation which is found in the field of free space electromagnetics [20] and more specifically in the mature field of antenna design for telecommunications [21]-[23]. In this field, the number of degrees of freedom in a coherent surface of physical area A_{phys} can be roughly assumed to be

$$N_{DoF}^{ant} \approx \frac{A_{phys}}{\left(\frac{\lambda}{2}\right)^2} \quad (23)$$

V. Conclusions

The use of Rytov's currents for the evaluation of the power radiated by thermally excited free electrons media is hampered by the problematic discretization, by the frequency dependence imposed, "Deus ex machina", to verify Planck's law. Here, we have proposed a procedure to identify the independent currents in a lossy medium: the degrees of freedom of the lossy medium.

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