Convex optimization for analysis of small antennas

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***Abstract*—A tutorial description of a method of moments implementation together with a Matlab toolbox for convex optimization to determine optimal current distributions on arbitrarily shaped antennas is presented. The optimal currents offer insight for antenna design and present performance bounds for antennas. Optimization formulations for maximal gain Q-factor quotient and antennas embedded in structures are discussed. The results are illustrated for planar geometries.**

***Index Terms*—physical bounds; convex optimization**

I. Introduction

The performance of antennas is limited by the electrical size (measured in wavelengths) of the antenna [1, 2]. Physical bounds express the trade-off between performance and size and are useful as they provide bounds solely based on the shape and size of the design volume. Chu [3] computed the stored and radiated energies outside a sphere circumscribing the antenna to determine a lower bound on the Q-factor, Q. It was generalized to arbitrary sized and shaped antennas in [4, 5] under the assumption of $Q \gg 1$, see also [6–9].

Here, convex optimization [10] is used as a tool to determine upper bounds on the antenna performance for many new cases [11]. In [11], we present results for minimum Q of superdirective antennas and minimum Q for antennas with a prescribed far field and show how antennas embedded in metallic structures can be included in the bounds. The formulation as convex optimization problems are advantageous as it offers very efficient solvers and explicit error estimates [10]. Convex optimization is often used in the synthesis of array patterns [10, 12].

II. Antenna parameters and MoM formulation

Following the approach in [11], we consider antennas in a volume V composed of non-magnetic materials with free space outside of V, see Fig. 1. The radiated field and stored energies are expressed in the antenna current $\boldsymbol{J}$ in V. We use a MoM approach to approximate the radiation vector and the stored energies [9, 11, 13]. We start with a method of moments (MoM) implementation of the electric field integral equation (EFIE) $\hat{\boldsymbol{n}}\times\mathcal{L}(\boldsymbol{J}) = \hat{\boldsymbol{n}}\times\boldsymbol{E}^{(\mathrm{i})}$. Expand the current density in local basis functions $\boldsymbol{J}(\boldsymbol{r}) \approx \sum_{n=1}^{N} J_n\boldsymbol{\psi}_n(\boldsymbol{r})$ and introduce the $N\times 1$ matrix $\mathbf{J}$ with elements J_n to simplify the notation. The basis functions are assumed to be divergence conforming and having vanishing normal components at the boundary [14].

The MoM solution of the electric field integral equation

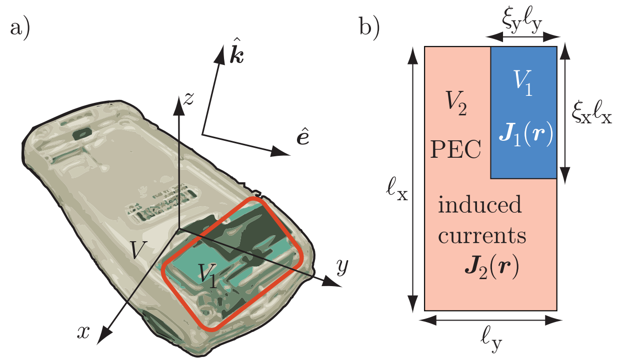

Fig. 1. a) Illustration of the radiating structure V and antenna region $V_1 \subset V$. b) planar rectangular region $V = V_1 \cap V_2$ with antenna region V_1 used in the numerical example.

(EFIE) has the matrix elements [14]

$$Z_{mn} = Z^{(\mathrm{m})}_{mn} - Z^{(\mathrm{e})}_{mn} = \mathrm{i}\int_{\partial V}\int_{\partial V}\Big(k^2\boldsymbol{\psi}_m(\boldsymbol{r}_1)\cdot\boldsymbol{\psi}_n(\boldsymbol{r}_2) - \nabla_1\cdot\boldsymbol{\psi}_m(\boldsymbol{r}_1)\nabla_2\cdot\boldsymbol{\psi}_n(\boldsymbol{r}_2)\Big)\frac{\mathrm{e}^{-\mathrm{j}k|\boldsymbol{r}_1-\boldsymbol{r}_2|}}{|\boldsymbol{r}_1-\boldsymbol{r}_2|}\,\mathrm{dS}_1\,\mathrm{dS}_2, \quad (1)$$

where we have decomposed the matrix elements into two parts and assumed surface currents. The $Z^{(\mathrm{m})}_{mn}$ and $Z^{(\mathrm{e})}_{mn}$ contain the $\boldsymbol{\psi}_m(\boldsymbol{r}_1)\cdot\boldsymbol{\psi}_n(\boldsymbol{r}_2)$ and $\nabla_1\cdot\boldsymbol{\psi}_m(\boldsymbol{r}_1)\nabla_2\cdot\boldsymbol{\psi}_n(\boldsymbol{r}_2)$ parts, respectively. Introduce a matrix $\mathbf{Z}^{(\mathrm{em})}$ with elements

$$Z^{(\mathrm{em})}_{mn} = \frac{k}{2}\int_{\partial V}\int_{\partial V}\Big(k^2\boldsymbol{\psi}_m(\boldsymbol{r}_1)\cdot\boldsymbol{\psi}_n(\boldsymbol{r}_2) - \nabla_1\cdot\boldsymbol{\psi}_m(\boldsymbol{r}_1)\nabla_2\cdot\boldsymbol{\psi}_n(\boldsymbol{r}_2)\Big)\mathrm{e}^{-\mathrm{j}k|\boldsymbol{r}_1-\boldsymbol{r}_2|}\,\mathrm{dS}_1\,\mathrm{dS}_2. \quad (2)$$

The radiation vector projected on $\hat{\mathbf{e}}$ defines the $N\times 1$ matrix $\mathbf{F}$ from

$$\hat{\mathbf{e}}^*\cdot\boldsymbol{F}(\hat{\boldsymbol{k}}) \approx \mathbf{F}^{\mathrm{H}}\mathbf{J} = \sum_{n=1}^{N} J_n\int_{\partial V}\hat{\mathbf{e}}^*\cdot\boldsymbol{\psi}_n(\boldsymbol{r})\mathrm{e}^{\mathrm{j}k\hat{\boldsymbol{k}}\cdot\boldsymbol{r}}\,\mathrm{dS}, \quad (3)$$

where the superscript, H, denotes the Hermitian transpose. The normalized stored electric energy [9, 11, 13, 15] is approximated as

$$w^{(\mathrm{e})}(\boldsymbol{J}) \approx \sum_{m=1}^{N}\sum_{n=1}^{N} J_m^* w^{(\mathrm{e})}_{mn} J_n = \mathbf{J}^{\mathrm{H}}\mathbf{X}_{\mathrm{e}}\mathbf{J}, \quad (4)$$

where the $N\times N$ matrix $\mathbf{X}_{\mathrm{e}} = \mathrm{Im}\{\mathbf{Z}^{(\mathrm{e})} + \mathbf{Z}^{(\mathrm{em})}\}$. The normalized stored magnetic energy is $w^{(\mathrm{m})}(\boldsymbol{J}) \approx \mathbf{J}^{\mathrm{H}}\mathbf{X}_{\mathrm{m}}\mathbf{J} = \mathbf{J}^{\mathrm{H}}(\mathrm{Im}\{\mathbf{Z}^{(\mathrm{m})} + \mathbf{Z}^{(\mathrm{em})}\})\mathbf{J}$. The matrices $\mathbf{X}_{\mathrm{e}}$ and $\mathbf{X}_{\mathrm{m}}$ are real-valued and symmetric. It is observed that $\mathbf{X}_{\mathrm{e}}$ can be indefinite for electrically large structures [9], so here we restrict the electrical size to be approximately less than half a wavelength.

III. CONVEX OPTIMIZATION

Here, maximization of G/Q for antennas is formulated as a convex optimization problem. The gain Q-factor quotient is

$$\frac{G(\hat{\boldsymbol{k}},\hat{\boldsymbol{e}})}{Q} = \frac{2\pi P(\hat{\boldsymbol{k}},\hat{\boldsymbol{e}})}{\mathrm{c}_0 k \max\{\widetilde{W}_{\mathrm{vac}}^{(\mathrm{e})},\widetilde{W}_{\mathrm{vac}}^{(\mathrm{m})}\}}. \tag{5}$$

We follow [9, 11] and scale $\boldsymbol{J}\to\alpha\boldsymbol{J}$ to reduce the problem to real-valued quantities $\hat{\boldsymbol{e}}^*\cdot\boldsymbol{F}\approx\mathbf{F}^{\mathrm{H}}\mathbf{J}$. Maximization of $P\sim|\mathbf{F}^{\mathrm{H}}\mathbf{J}|^2$ is replaced by maximization of $\mathrm{Re}\{\mathbf{F}^{\mathrm{H}}\mathbf{J}\}$. This gives the convex optimization problem

$$\begin{aligned} &\text{maximize} && \mathrm{Re}\{\mathbf{F}^{\mathrm{H}}\mathbf{J}\}\\ &\text{subject to} && \mathbf{J}^{\mathrm{H}}\mathbf{X}_{\mathrm{e}}\mathbf{J}\leq 1\\ &&& \mathbf{J}^{\mathrm{H}}\mathbf{X}_{\mathrm{m}}\mathbf{J}\leq 1. \end{aligned} \tag{6}$$

This is a quadratically constrained linear program (QCLP) that can be solved with the Matlab toolbox CVX [16]

```
cvx_begin
  variable J(n) complex;
  dual variables We Wm
  maximize(real(F'*J))
  subject to
    We: quad_form(J,Xe) <= 1;
    Wm: quad_form(J,Xm) <= 1;
cvx_end
```

IV. NUMERICAL EXAMPLE

We consider an antenna structure confined to a planar rectangle, see Fig. 1b. The region V_2 is assumed to be PEC and is fixed, *i.e.*, the antenna designer cannot change the material properties in V_2 but can use it as a radiator by inducing currents on it. The antenna designer is assumed to use the region V_1 to design the antenna, here modeled by assuming arbitrary currents in V_1. The results for $\xi_{\mathrm{x}}=1$ and $\xi_{\mathrm{y}}=\{0.1,0.2,1\}$ are depicted. It is observed that the performance deteriorates as $\ell_{\mathrm{x}}/\lambda\to 0$ and $\ell_{\mathrm{y}}\to 0$. There is also a region $\ell_{\mathrm{x}}/\lambda\to 0.4$, where the performance for $\xi_{\mathrm{y}}=\{0.1,0.2\}$ is close to the $\xi_{\mathrm{y}}=1$ case, where the entire rectangle is used to design the antenna

V. CONCLUSIONS

It shown that convex optimization can be used to determine optimal currents on antenna structures and physical bounds for antennas confined to the antenna structure. The results are illustrated for an antenna region embedded next to a finite ground plane.

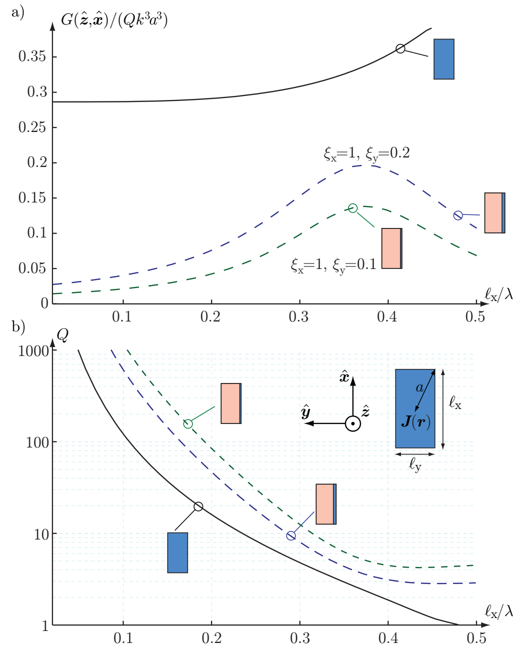

Fig. 2. Physical bound on G/Q for antennas confined to the region V_1 in Fig. 1b. The region V is a planar rectangle with side lengths ℓ_{x} and $\ell_{\mathrm{y}}=\ell_{\mathrm{x}}/2$. a) bound on G/Q. b) resulting Q for lossless structures.

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